Some examples of open manifolds with positive Ricci curvature

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Part 1. In a nutshell

Past constructions

Question

For open (complete and non-compact) manifolds, what are the differences between nonnegative/positive sectional curvature and nonnegative/positive Ricci curvature?

There are lots of examples addressing the above question (e.g., constructions by Sha-Yang, Menguy, Perelman, etc.)
For this talk, we look at Nabbonand's example and some related constructions.

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Nabbonand's examples (1980): $\mathbb{R}^k \times S^1$ ($k \geq 3$) admits a Riemannian metric with $\mathrm{Ric} > 0$.
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Wei's examples (1988): $\mathbb{R}^k \times N$, where N is a nilmanifold and k large, admits a Riemannian metric with $\mathrm{Ric} > 0$.
Cf: If M has $\sec \geq 0$, then $\pi_1(M)$ is virtually abelian.

New observations

By investigating the universal covers of related constructions, we give negative answers to the following open problems (Pan-Wei 2021, 2022).

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Open Problem

For an open manifold with nonnegative Ricci curvature, is it true that its Busemann function at a point must be proper?

Open Problem

For a Ricci limit space, is it true that the Hausdorff dimension of its singular set cannot exceed that of the regular set?

Both are true under the sectional curvature condition.

Part 1. In a nutshell

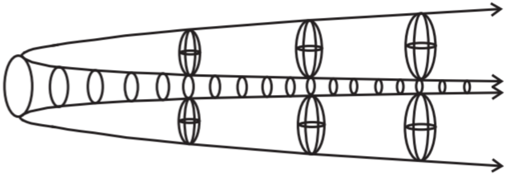
Part 2. Past constructions

Nabonnard/Bergery's examples (1980/1986)

Let (N, g_0) be a closed manifold of $\mathrm{Ric}(g_0) \geq 0$. Doubly warped product on $\mathbb{R}^k \times N$.

$$M = [0, \infty) \times_f S^{k-1} \times_h N, \quad dr^2 + f^2(r) ds^2 + h^2(r) g_0$$

has $\mathrm{Ric} > 0$ for suitable (implicit) $f(r)$, $h(r)$, and large k.
f odd, h even, $f'(0) = 1$, $h(0) > 0$, ...
$h(r)$ decreases $\to 0$ or $\to c > 0$ as $r \to +\infty$.



Wei's examples (1988)

Let $\widetilde{N}$ be a simply connected nilpotent Lie group and let Γ be a lattice in $\widetilde{N}$. $N = \widetilde{N}/\Gamma$ admits a family of metrics $\{g_r\}_{r\geq 0}$ with

$$|\sec(g_r)| \leq \frac{c}{1+r^2}.$$

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$$|\mathrm{sec}(g_r)| \leq \frac{c}{1+r^2}.$$

Then for a suitable function $f(r)$, the warped product

$$M = [0,\infty) \times_f S^{k-1} \times N_r, \quad dr^2 + f^2(r)ds^2 + g_r$$

has $\mathrm{Ric} > 0$ when p is large.
$\mathrm{diam}(N_r)$ decays at a rate of $r^{-\alpha}$ $(\alpha > 0)$.
$\pi_1(M) = \pi_1(N)$ is not virtually abelian.

Choice of warping function h

Doubly warped product

$$M = [0,\infty) \times_f S^{k-1} \times_h S^1, \quad dr^2 + f^2(r)ds^2 + h^2(r)g_0$$

with $\mathrm{Ric} > 0$.

Wei's construction, a polynomial decay h:

$$f(r) = r(1+r^2)^{-1/4}, \quad h(r) = (1+r^2)^{-\alpha}.$$

A logarithm decay h:

$$f(r) = \frac{\sqrt{\ln 2}\cdot r}{\ln^{1/2}(2+r^2)}, \quad h(r) = \ln^{-\alpha}(2+r^2).$$

Let M be an open Riemannian manifold.

Busemann function associated with a unit speed ray c:

$$b_c(x) = \lim_{t\to+\infty} t - d(x, c(t)).$$

We can view $b_c(x)$ as (the negative of) a renormalized distance from x to infinity along c.

3.1 Busemann function

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Busemann function at a point $p \in M$:

$$b_p(x) = \sup_c b_c(x),$$

where sup is taken over all rays starting at p.

For convenience, we use the term *Busemann function* to refer to the one at some point.

Theorem (Cheeger-Gromoll 1970)

If M has $\sec \geq 0$, then the Busemann function is convex and proper.

Remark: This is one of the key steps in the proof of Cheeger-Gromoll's soul theorem...also, properness follows from convexity.

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Theorem (Cheeger-Gromoll 1970)

If M has $\mathrm{Ric} \geq 0$, then the Busemann function is subharmonic.

Remark: This is one of the key steps in the proof of Cheeger-Gromoll's splitting theorem.

Is it proper?

Open Problem (since the time of splitting theorem, per private communication with Cheeger):

Question

Is the Busemann function always proper when M has $\mathrm{Ric} \geq 0$?

Remark: The properness of b_p does not depend on the choice of $p \in M$.

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Theorem

Let M be an open manifold with $\mathrm{Ric} \geq 0$. Then the Busemann function is proper when
(1) M has Euclidean volume growth (Shen 1996), or
(2) M has linear volume growth (Sormani 1998).

Answer

Theorem (Pan-Wei 2022)

Given any integer $n \geq 4$, there is an open n-manifold with positive Ricci curvature and a non-proper Busemann function.

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Question: What if we also assume that all asymptotic cones of M are polar, is the Busemann function proper?

Theorem (Pan-Wei 2022)

There is an open manifold with positive Ricci curvature, a unique polar asymptotic cone, and a non-proper Busemann function.

Look at the covers

Bottom-line:

We will use the Riemannian universal cover $\widetilde{M}$ of a doubly warped product $M = [0, \infty) \times_f S^{k-1} \times_h S^1$ with suitable functions f and h as our examples.

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When h decreases to 0 as $r \to \infty$, $\widetilde{M}$ has a non-proper Busemann function.
When h has logarithm decay, $\widetilde{M}$ has a unique polar asymptotic cone, as the standard half-plane, and a non-proper Busemann function.

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To prove the Busemann function at a chosen base point $\tilde{p}$ is non-proper, we need to find all the rays in $\widetilde{M}$ starting at $\tilde{p}$ ...firstly find all geodesics in M starting at p.

Geodesics in M

$M = [0,\infty) \times_f S^{k-1} \times_h S^1$...write each point in M as (r,x,y)...fix a point $p = (0,x,y)$ at $r = 0$ as our base point.

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For each $x \in S^{k-1}$, we define a submanifold
$C(x) = \{(r, \pm x, y) | r \geq 0, y \in S^1\}$.
It is totally geodesic and geodesically complete.
The induced Riemannian metric on $C(x)$ is isometric to the surface revolution $\mathbb{R} \times_h S^1$.

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(By classical differential geometry and our choice of h) a geodesic in $\mathbb{R} \times_h S^1$ starting at $p = (0,y)$ has one of the following forms:
(1) a radial ray,
(2) a closed geodesic as the circle $\{r = 0\}$,
(3) a bounded geodesic oscillating between $\{r \geq 0\}$ and $\{r \leq 0\}$ indefinitely.

Let c be a unit speed ray $\widetilde{M}$ starting at $\tilde{p}$.

(By the above classification and the splitting theorem,) c must project to a radial ray $\bar{c}(t) = (r(t), x, y)$ in M for some $x \in S^{k-1}$ and $y \in S^1$.

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Let γ be a generator of $\Gamma = \pi_1(M, p) \simeq \mathbb{Z}$. For all $l \in \mathbb{Z}$:
$d(\gamma^l \tilde{p}, c(t)) \geq d(p, \bar{c}(t)) = t; \quad d(\gamma^l \tilde{p}, c(t)) \leq t + |l| \cdot 2\pi h(t)$.
$\Rightarrow b_c(\gamma^l \tilde{p}) = 0$.

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$\Rightarrow b_c(\gamma^l \tilde{p}) = 0.$

$\Rightarrow b_{\tilde{p}}(\gamma^l \tilde{p}) = \sup_c b_c(\gamma^l \tilde{p}) = 0.$
Thus the level set $b_{\tilde{p}}^{-1}(0)$ contains the entire orbit $\Gamma\tilde{p}$, which is unbounded. That's it.

3.2 Ricci limit spaces

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Gromov's precompactness theorem

Let (M_i, p_i) be a sequence of complete Riemannian n-manifolds of $\mathrm{Ric} \geq -(n-1)$, then (M_i, p_i) has a Gromov-Hausdorff convergent subsequence with limit as a length metric space (X, p).

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For a Ricci limit space X, a point $x \in X$ is called

- *k-regular*, if every tangent cone at x is isometric to $\mathbb{R}^k$.
- *singular*, if otherwise.

$\mathcal{R}^k$: the set of k-regular points.
$\mathcal{S}$: the set of singular points.

Rectifiable dimension (Cheeger-Colding 2000, Colding-Naber 2012)

Let X be a Ricci limit space. There is a unique integer $k \in [0, n]$ such that (among many other properties,) $\mathcal{R}^k$ has full ν-measure for any renormalized limit measure ν. Moreover, $\dim_H(\mathcal{R}^k) = k$.

This integer k is called the *rectifiable dimension* of X.

Regularity theory of Ricci limit spaces

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Question (Cheeger-Colding 1996)

For a collapsing Ricci limit space, is it true that the Hausdorff dimension of the singular set cannot exceed that of the regular set? Equivalently, Rectifiable dimension=Hausdorff dimension?

Answer

Theorem (Pan-Wei 2021)

Given any $\beta > 0$, for n sufficiently large (depending on β), there is an open n-manifold N with $\mathrm{Ric} > 0$, whose asymptotic cone (Y, y) satisfies the following:
(1) $Y = \mathcal{S} \cup \mathcal{R}^2$, where $\mathcal{S}$ is the singular set and $\mathcal{R}^2$ is the set of 2-regular points;
(2) $\mathcal{S}$ has Hausdorff dimension $1 + \beta$.

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When $\beta > 1$, $\dim_H Y = \dim_H \mathcal{S} > \dim_H \mathcal{R}$.
Limit measure and any dimensional Hausdorff measure may not be mutually absolutely continuous.

Asymptotic Cones

Let (M, p) be an open manifold of $\mathrm{Ric} \geq 0$. For any $r_i \to \infty$, passing to a subsequence if necessary, we obtain the pointed Gromov-Hausdorff convergence:

$$(r_i^{-1} M, p) \xrightarrow{GH} (Z, z).$$

We call (Z, z) an *asymptotic cone* of M.

Example: The asymptotic cone of a flat cylinder is a line.

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Fact: When M has $\sec \geq 0$, its asymptotic cone (Z, z) is unique as a metric cone with vertex z.

In general, when $\mathrm{Ric} \geq 0$, its asymptotic cone may not be unique and may not be polar at the base point.

For the universal cover $(\widetilde{M},\tilde{p})$ with $\Gamma=\pi_1(M,p)$-action, we can obtain the equivariant pointed GH convergence:

$$\begin{array}{ccc}
(r_i^{-1}\widetilde{M},\tilde{p},\Gamma) & \xrightarrow{GH} & (Y,y,G) \\
\downarrow \pi & & \downarrow \pi \\
(r_i^{-1}M,p) & \xrightarrow{GH} & (Z,z)=(Y/G,\bar{y}),
\end{array}$$

where G is a closed subgroup of $\mathrm{Isom}(Y)$. We call (Y,y,G) an *equivariant asymptotic cone* of $(\widetilde{M},\Gamma)$.

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Example: Flat cylinder...

Fact: When M has $\sec \geq 0$, equivariant asymptotic cone of $(\widetilde{M}, \Gamma)$ is unique as $(C(X), v, G)$, where Gv is isometric to an $\mathbb{R}^k$-factor.

Bottom-line:
We will use the Riemannian universal cover $\widetilde{M}$ of a doubly warped product $M = [0,\infty) \times_f S^{k-1} \times_h S^1$ with suitable functions f and h as our examples.

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For any equivariant asymptotic cone (Y, y, G) of $(\widetilde{M}, \Gamma)$, Y has rectifiable dimension 2; the orbit Gy is the singular set and is homeomorphic to $\mathbb{R}$.
When h has polynomial decay $\sim r^{-\beta}$, the orbit Gy has Hausdorff dimension $1 + \beta$.

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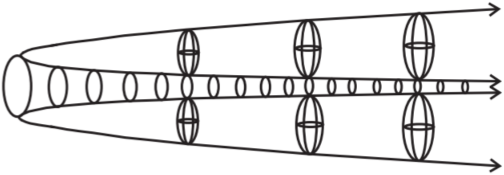
To prove this, we need to study the $\pi_1(M, p)$-action on $\widetilde{M}$.

Distance estimate

$f(r) = r(1+r^2)^{-1/4}$, $h(r) = (1+r^2)^{-\alpha}$ as in Wei's construction.
Let $p \in M$ at $r = 0$ and γ be a generator of $\pi_1(M,p) = \mathbb{Z}$.
Let c_l be a minimal representing loop of γ^l.



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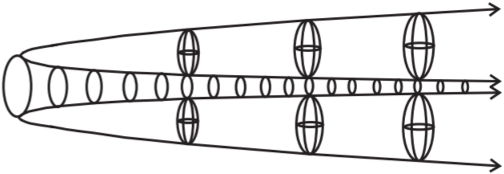

Because h is decreasing, c_l goes further into the end as l increases; this also shortens the length of c_l.
A simple, but crucial, estimate: As $l \to \infty$,

- $\mathrm{length}(c_l) \sim l^{\frac{1}{1+2\alpha}}$,
- $\mathrm{size}(c_l) \sim l^{\frac{1}{1+2\alpha}}$.

Asymptotic cones

$$\begin{tikzcd}
(r_i^{-1}\widetilde{M}, \tilde{p}, \Gamma) \arrow[r, "GH"] \arrow[d, "\pi"] & (Y, y, G) \arrow[d, "\pi"] \\
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Recall that $f(r) \sim \sqrt{r}$ and $h(r) \sim r^{-2\alpha}$, thus Z is a half-line $[0, \infty)$ with $z = 0$.

$Z = Y/G$...Y as attaching orbits to $Z = [0, \infty)$...$G = \mathbb{R}$ and Y is homeomorphic to $\mathbb{R} \times [0, \infty)$.

Blank space for drawing:

The orbit Gy

Let $g \in G$ with $d(gy, y) = 1$. Then we can naturally assign every point in Gy a coordinate $(b, 0)$, where $b \in \mathbb{R}$, such that $g^b y$ corresponds to $(b, 0)$.

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The (extrinsic) distance on $Gy = \mathbb{R} \times \{0\}$ satisfies
$C_1 \cdot |b_1 - b_2|^{\frac{1}{1+2\alpha}} \leq d((b_1, 0), (b_2, 0)) \leq C_2 \cdot |b_1 - b_2|^{\frac{1}{1+2\alpha}}$
for all $b_1, b_2 \in \mathbb{R}$, where C_1, $C_2 > 0$ only depend on α.

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for all $b_1, b_2 \in \mathbb{R}$, where C_1, $C_2 > 0$ only depend on α.

Gy is not geodesic; in fact, the intrinsic distance between any two distinct points in Gy is infinity.

(Gy, d) has Hausdorff dimension $1 + 2\alpha$.

Some remarks

With some further estimates, we can show that any point $z \in Y - Gy$ is 2-regular.

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As $\alpha \to 0$ in $h(r) = (1 + r^2)^{-\alpha}$, the eGH distance between (Y, y, G) and $(\mathbb{R} \times [0, \infty), (0, 0), \mathbb{R})$ tends to 0, where $\mathbb{R}$ acts as translations on $\mathbb{R} \times [0, \infty)$.

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For any asymptotic cone (Y, y) of $(\widetilde{M}, \tilde{p})$, Y is non-polar at y. The first examples of non-polar asymptotic cones are constructed by Menguy.

As $\alpha \to 0$ in $h(r) = (1 + r^2)^{-\alpha}$, the eGH distance between (Y, y, G) and $(\mathbb{R} \times [0, \infty), (0, 0), \mathbb{R})$ tends to 0, where $\mathbb{R}$ acts as translations on $\mathbb{R} \times [0, \infty)$.

If $h(r)$ has logarithm decay $\sim \ln^{-\alpha}(r)$ or it $\to c > 0$ as $r \to \infty$, then (Y, y) splits isometrically as $\mathbb{R} \times [0, \infty)$. Thus logarithm decaying h gives examples with polar asymptotic cones and non-proper Busemann functions.